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# Finite-dimensional representations of $U_q(C(n+1))$ at arbitrary $q$

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**Abstract.** A method is developed to construct irreducible representations (irreps) of the quantum supergroup $U_q(C(n+1))$ in a systematic fashion. It is shown that every finite-dimensional irrep of this quantum supergroup at generic $q$ is a deformation of a finite-dimensional irrep of its underlying Lie superalgebra $C(n+1)$ and is essentially uniquely characterized by a highest weight. The character of the irrep is given. When $q$ is a root of unity, all irreps of $U_q(C(n+1))$ are finite-dimensional; multiply atypical highest-weight irreps and (semi)cyclic irreps also exist. As examples, all the highest weight and (semi)cyclic irreps of $U_q(C(2))$ are thoroughly studied.

## 1. Introduction

This is the third of a series of papers which systematically develop the representation theory of the quantum supergroups [1] associated with the basic classical Lie superalgebras [2]. The first paper [3] studied the structures of the finite-dimensional representations of the quantum supergroup $U_q(gl(m \mid n))$ at arbitrary $q$ (the finite-dimensional irreps of $U_q(gl(m \mid 1))$ have been explicitly constructed in [4]), while the second one [5] treated the representation theory of $U_q(B(0,n))$. It is the aim of the present paper to study $U_q(C(n+1))$.

Vigorous study of the theory of Lie superalgebras began in the 1970s (for reviews see [2]), largely motivated by the discovery of supersymmetry in theoretical physics. It was clear from the very beginning that although some properties of ordinary Lie algebras are shared by their $\mathbf{Z}_2$-graded counterparts, the Lie superalgebras are by no means straightforward generalizations of ordinary Lie algebras; in particular, their representation theory is drastically different from that of the latter.

Recall that the Weyl groups are of paramount importance in the study of the finite-dimensional irreps of the Lie algebras; they enable one to compute the characters which embody all the information about the weight spaces and dimensions, etc, of the irreps. Also, the finite-dimensional representations of Lie algebras are completely reducible. This fact makes it possible to understand the structures of all finite-dimensional representations.

However, it is not possible in general (except for $osp(1 \mid 2n)$) to introduce a Weyl group for a Lie superalgebra, which is not simply the Weyl group of the maximal even subalgebra. The so-called Weyl groups of Lie superalgebras embody little useful information about the odd generators, thus not allowing the determination of the structures of irreps. Also, finite-dimensional representations of Lie superalgebras are not completely reducible. These facts make the representation theory of Lie superalgebras an extremely difficult subject to study.

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Quantum supergroups [1] are one-parameter deformations of the universal enveloping algebras of basic classical Lie superalgebras, the origin of which may be traced back to the Perk–Schultz solution of the Yang–Baxter equation and the work of Bazhanov and Shadrikov [6], although the systematical study of these algebraic structures only began very recently [1]. It has become clear that the quantum supergroups are of great importance to a variety of fields in theoretical physics and mathematics, e.g., quantum field theory and knot theory, apart from soluble models in statistical mechanics. In all the applications of quantum supergroups, their finite-dimensional representations play a central role. However, our understanding of such representations is very incomplete.

It is by now well known that the representation theory of ordinary quantum groups at generic $q$ is very much the same as that of the corresponding Lie algebras [7]. Lusztig and Rosso [7] proved that each finite-dimensional irrep of a quantum group is a deformation of an irrep of the underlying Lie algebra, and all finite-dimensional representations are completely reducible. Rosso's proof made essential use of the properties of the Weyl groups of the underlying Lie algebras, while the main ideas of Lusztig's proof are as follows. In the $q \to 1$ limit, an integrable highest-weight irrep $\pi$ of a quantum group $U_q(g)$ reduces to an indecomposable representation $\tilde{\pi}$ of its underlying Lie algebra $g$. As integrable representations of ordinary Lie algebras are completely reducible, $\tilde{\pi}$, being indecomposable, must be an irreducible representation of $g$.

Obviously none of these proofs can generalize to quantum supergroups, as the basic ingredients, namely Weyl groups and complete reducibility of integrable representations, are lacking in this case. In view of Lusztig's work, it even appears possible intuitively that a finite-dimensional irrep of a quantum supergroup at generic $q$ may reduce to an indecomposable but reducible representation of the underlying Lie superalgebra in the limit $q \to 1$. Fortunately it turned out not to be the case, at least for $U_q(gl(m \mid n))$ and $U_q(osp(1 \mid 2n))$, as shown in [3] and [5].

One of the main results of the present paper is the proof that every finite-dimensional irrep of the quantum supergroup $U_q(C(n+1))$ at generic $q$ reduces to an irrep of the underlying Lie superalgebra $C(n+1)$, and the two irreps have the same weight space decomposition. This result enables us to gain a thorough understanding of the structures of finite-dimensional irreps of $U_q(C(n+1))$, in particular, to write down their character formula, as $C(n+1)$ happens to be one of the very few Lie superalgebras having a well developed representation theory [8].

When $q$ is a root of unity, we will develop a method allowing us to construct $U_q(C(n+1))$ irreps in a systematic fashion. The representation theory in this case changes dramatically; in particular, all irreps are finite dimensional, and (semi)cyclic irreps and multiply atypical irreps appear.

The arrangement of the paper is as follows. In section 2, we prove a generalized BPW theorem for $U_q(C(n+1))$. In section 3 we generalize Kac's induced module construction for Lie superalgebras to this quantum supergroup at arbitrary $q$, and also thoroughly investigate the structures of the finite-dimensional irreps at generic $q$. In section 4 we explicitly construct all the irreps of $U_q(C(2))$ using the general theory developed in the earlier sections.

## 2. BPW theorem for $U_q(C(n+1))$

This section studies the structure of the $\mathbb{Z}_2$ graded Hopf algebra $U_q(C(n+1))$. In particular, a generalized BPW theorem for this quantum supergroup will be proved, and an explicit basis for it will also be constructed. Results of this section will be repeatedly applied throughout the paper.

### 2.1. *Definition of $U_q(C(n+1))$*

Let us begin by defining the quantum supergroup $U_q(C(n+1))$. Recall that Lie superalgebras admit different simple root systems, which cannot be mapped onto one another by the Weyl groups of their maximal even subalgebras. As quantization treats the Cartan and simple generators differently from the rest, it is not clear whether the quantum supergroups obtained by quantizing the same Lie superalgebra but using different simple root systems are algebraically equivalent (It is not difficult to convince oneself by examining simple examples that the resultant quantum supergroups are co-algebraically inequivalent). However, we will not be concerned with this problem here. Instead we merely take $U_q(C(n+1))$ to be the quantization of the universal enveloping algebra of the type I superalgebra $C(n+1)$ with the commonly used simple root system, namely, the one with a unique odd simple root.

To describe this simple root system, we introduce the $(n+1)$-dimensional Minkowski space $H^*$ with a basis $\{\delta_i \mid i = 0, 1, 2, \ldots, n\}$ and the bilinear form $(\ ,\ ): H^* \times H^* \to C$ defined by

$$(\delta_i,\ \delta_j) = -(-1)^{\delta_{0i}} \qquad \forall i, j.$$

Then, following Kac, the simple roots of $C(n+1)$ can be expressed as

$$\begin{aligned} &\alpha_i = \delta_i - \delta_{i+1} \qquad i = 0, 2, \ldots, n-1 \\ &\alpha_n = 2\delta_n \end{aligned}$$

where $\alpha_0$ is the unique odd simple root. A convenient version of the Cartan matrix $A = (a_{ij})_{i,j=0}^n$ for $C(n+1)$ is

$$\begin{aligned} &a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \qquad \forall i > 0 \\ &a_{0j} = (\alpha_0, \alpha_j). \end{aligned}$$

We denote by $\Delta_0^+$ and $\Delta_1^+$ the set of the even positive roots and that of the odd positive roots of $C(n+1)$ respectively. Then

$$\begin{aligned} &\Delta_0^+ = \{\delta_i - \delta_j, \delta_i + \delta_j\ 2\delta_i \mid 0 < i < j\} \\ &\Delta_1^+ = \{\delta_0 \pm \delta_i \mid i > 0\}. \end{aligned}$$

For later use, we also define

$$\begin{aligned} &\rho_\theta = \tfrac{1}{2} \sum_{\alpha \in \Delta_\theta^+} \alpha \qquad \theta = 1, 2 \\ &\rho = \rho_0 - \rho_1. \end{aligned}$$

Let $q$ be an indeterminate, and define

$$q_i = \begin{cases} q^{(\alpha_i, \alpha_i)/2} & i > 0 \\ q & i = 0. \end{cases}$$

The quantum supergroup $U_q(C(n+1))$ is the unital $\mathbf{Z}_2$-graded algebra on $\mathbf{C}[q, q^{-1}]$, which is generated by $\{e_i,\ f_i\ K_i^{\pm} \mid i = 0, 1, \ldots, n\}$ with the relations

$$\begin{aligned} &K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1} \\ &K_i K_i^{-1} = 1 \\ &K_i e_j K_i^{-1} = q_i^{a_{ij}} e_j \\ &K_i f_j K_i^{-1} = q_i^{-a_{ij}} f_j \\ &[e_i, f_j\} = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1})(e_0)^2 = (f_0)^2 = 0 \end{aligned}$$

$$\sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_{q_i} e_i^{1-a_{ij}-\mu} e_j e_i^\mu = 0 \qquad i \neq 0$$
$$\sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_{q_i} f_i^{1-a_{ij}-\mu} f_j f_i^\mu = 0 \qquad i \neq 0 \tag{1}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!} \qquad m \geqslant n$$
$$[k]_q! = \begin{cases} \prod_{i=1}^{k} \frac{q^i - q^{-i}}{q - q^{-1}} & k > 0 \\ 1 & k = 0. \end{cases}$$

In (1), $[x, y\} = xy - (-1)^{[x][y]} yx$, with the gradation indices $[x]$ and $[y]$ defined by

$$[K_i] = 0 \qquad \forall i \qquad [e_i] = [f_i] = \begin{cases} 0 & i > 0 \\ 1 & i = 0 \end{cases}$$

and for any $u, v$ which are monomials in $e_i, f_i, K_i^{\pm 1}$, $i = 0, 1, \ldots, n$, $[uv] \equiv [u] + [v] (\bmod 2)$. This makes $U_q(C(n+1))$ a $\mathbf{Z}_2$-graded algebra with $U_q(C(n+1)) = U_0 \oplus U_1$, where $U_0 = \{u \in U_q(g) \mid [u] = 0\}$, $U_1 = \{u \in U_q(g) \mid [u] = 1\}$. We will call the elements of $U_0$ even and those of $U_1$ odd. We also associate with their product $uv$ an element of $H^*$, $wt(uv) = wt(u) + wt(v)$, called the weight, such that $wt(e_i) = -wt(f_i) = \alpha_i$, $wt(K_i^{\pm 1}) = 0$. If $w \in U_q(C(n+1))$ is a linear combination of monomials of the same weight $\mu \in H^*$, we define $wt(w) = \mu$.

The quantum supergroup $U_q(C(n+1))$ has the structures of a $\mathbf{Z}_2$ graded Hopf algebra with invertible antipode. One consistent co-multiplication reads

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}$$
$$\Delta(e_i) = e_i \otimes 1 + K_i \otimes e_i$$
$$\Delta(f_i) = f_i \otimes K_i^{-1} + 1 \otimes f_i$$

and the corresponding co-unit $\epsilon$ and antipode $S$ are, respectively, given by

$$\epsilon(e_i) = \epsilon(f_i) = 0$$
$$\epsilon(K_i) = \epsilon(K_i^{-1}) = 1$$
$$S(e_i) = -K_i^{-1} e_i$$
$$S(f_i) = -f_i K_i$$
$$S(K_i^{\pm 1}) = K_i^{\mp 1} \qquad \forall i.$$

Note that $\{e_i, f_i, K_i^{\pm} \mid i = 1, 2, \ldots, n\}$ generate a subalgebra $U_q(sp(2n)) \subset U_q(C(n+1))$. Together with $\{K_0^{\pm 1}\}$, they generate $U_q(sp(2n) \oplus u(1))$ which we will refer to as the maximal even quantum subgroup of $U_q(C(n+1))$.

### 2.2. $U_q(C(n+1))$ at generic $q$

In order to study the structures of $U_q(C(n+1))$, we introduce the $\mathbf{Z}_2$ graded automorphism

$$\varpi(e_i) = f_i \qquad \varpi(f_i) = e_i \qquad \varpi(K_i) = K_i$$
$$\varpi(q) = q^{-1} \qquad \varpi(c) = c^* \qquad c \in \mathbf{C}$$

and the anti-automorphism

$$\omega(e_i)=f_i \qquad \omega(f_i)=e_i \qquad \omega(K_i)=K_i^{-1}$$
$$\omega(q)=q^{-1} \qquad \omega(c)=c^* \qquad c\in\mathbf{C}$$

which are also required to satisfy $\varpi(uv)=(-1)^{[u][v]}\varpi(u)\varpi(v)$, $\omega(uv)=\omega(v)\omega(u)$, for any homogeneous elements $u,v\in U_q(C(n+1))$, and generalize to all elements of $U_q(C(n+1))$ through linearity.

Define the maps $T_i: U_q(C(n+1))\to U_q(C(n+1))$, $i=1,2,\ldots,n$, by

$$\begin{aligned}
T_i(K_j)&=K_jK_i^{-a_{ij}} \qquad \forall j\\
T_i(e_i)&=-f_iK_i\\
T_i(f_i)&=-K_i^{-1}e_i\\
T_i(e_j)&=\sum_{t=0}^{-a_{ij}}(-1)^{t-a_{ij}}q_i^{-t}\frac{(e_i)^{-a_{ij}-t}e_j(e_i)^t}{[-a_{ij}-t]_{q_i}![t]_{q_i}!}\\
T_i(f_j)&=\sum_{t=0}^{-a_{ij}}(-1)^{t}q_i^{-a_{ij}-t}\frac{(f_i)^{-a_{ij}-t}f_j(f_i)^t}{[-a_{ij}-t]_{q_i}![t]_{q_i}!} \qquad \forall j\neq i.
\end{aligned}$$

Then

*Lemma 1.* The $T_i$, $i=1,2,\ldots,n$, define algebra automorphisms of $U_q(C(n+1))$, thus generating a group which will be denoted by $\widehat{W}$. They also satisfy

$$\begin{aligned}
&T_i\omega=\omega T_i\\
&T_i^{-1}=\varpi T_i\varpi^{-1}.
\end{aligned}\tag{2}$$

*Proof.* Restricted to the maximal even quantum subgroup $U_q(sp(2n)\oplus u(1))$, the $T_i$s coincide with the Lusztig automorphisms [9] of this quantum group. Thus we only need to show that $T_1$ preserves the relations in (1) involving $e_0$ and $f_0$, in order to prove that $T_i$s are algebra homomorphisms of $U_q(C(n+1))$, since $T_1$ is the only map amongst the $T_i$s which acts non-trivially on $e_0$ and $f_0$. Consider, say, $\{T_1(e_0),T_1(f_0)\}$ when $n>1$. Now,

$$\begin{aligned}
T_1(e_0)&=-e_1e_0+qe_0e_1\\
T_1(f_0)&=-f_0f_1+q^{-1}f_1f_0.
\end{aligned}$$

Simple calculations lead to

$$\begin{aligned}
\{T_1(e_0),T_1(f_0)\}&=\frac{K_0K_1-K_0^{-1}K_1^{-1}}{q-q^{-1}}\\
&=T_1\left(\frac{K_0-K_0^{-1}}{q-q^{-1}}\right).
\end{aligned}$$

The other relations can be checkea similarly. Equation (2) can be proved by explicitly working out the actions of the maps involved on the simple and Cartan generator of $U_q(C(n+1))$.

The maximal even quantum subgroup $U_q(sp(2n)\oplus u(1))$ admits the following decomposition

$$U_q(sp(2n)\oplus u(1))=B_-B_0B_+$$

where $B_+$ is generated by $\{e_i \mid i>0\}$, $B_-$ by $\{f_i \mid i>0\}$, and $B_0$ by $\{K_i^{\pm 1}, \mid i=0,1,\ldots,n\}$. A basis for $B_0$ is given by $\{K^{(\hat r)}H^{(\hat s)} \mid \hat r,\hat s\in\mathbf{Z}_+^{n+1}\}$, with $\hat r=(r_0,r_1,\ldots,r_n)$, $K^{(\hat r)}=\prod_{i=0}^n K_i^{r_i}$, $H^{(\hat r)}=\prod_{i=0}^n((K_i-K_i^{-1})/(q_i-q_i^{-1}))^{r_i}$.

Following [10], we construct bases for $B_+$ and $B_-$ by considering the maximal element $T$ of $\widehat{W}$, a reduced expression for which reads

$$T = T_{i_1}T_{i_2}\dots T_{i_{n^2}}$$
$$= (T_1T_2\dots T_{n-1}T_nT_{n-1}\dots T_2T_1)(T_2\dots T_{n-1}T_nT_{n-1}\dots T_2)\dots(T_{n-1}T_nT_{n-1})T_n.$$

We define

$$\begin{aligned}
E_{\beta_1} &= e_1 \\
F_{\beta_1} &= f_1 \\
E_{\beta_t} &= T_{i_1}T_{i_2}\dots T_{i_{t-1}}(e_{i_t}) \\
F_{\beta_t} &= T_{i_1}T_{i_2}\dots T_{i_{t-1}}(f_{i_t}) \qquad t = 1, 2, \dots, n^2
\end{aligned}$$

where $\beta_t \in \Delta_0^+$, and clearly $F_{\beta_t} = \omega(E_{\beta_t})$. Also observe the following important facts [10]: $E_{\beta_t} \in B_+$, $F_{\beta_t} \in B_-$, and

$$\begin{aligned}
&\{E^{(\hat{k})} = (E_{\beta_1})^{k_1}(E_{\beta_2})^{k_2}\dots(E_{\beta_{n^2}})^{k_{n^2}} \mid \hat{k} \in \mathbf{Z}_+^{n^2}\} \\
&\{F^{(\hat{k})} = (F_{\beta_{n^2}})^{k_{n^2}}(F_{\beta_{n^2-1}})^{k_{n^2-1}}\dots(F_1)^{k_1} \mid \hat{k} \in \mathbf{Z}_+^{n^2}\}
\end{aligned} \tag{3}$$

form bases for $B_+$ and $B_-$ respectively.

To study the odd elements of $U_q(C(n+1))$, we define

$$\begin{aligned}
\psi_1 &= e_0 \\
\psi_{i+1} &= \psi_i e_i - q e_i \psi_i \qquad 1 \leqslant i < n \\
\psi_{-n} &= \psi_n e_n - q^2 e_n \psi_n \\
\psi_{-i} &= \psi_{-i-1} e_i - q e_i \psi_{-i-1} \qquad 0 < i < n \\
\phi_{\pm i} &= \omega(\psi_{\pm i}) \qquad i = 1, 2, \dots, n.
\end{aligned} \tag{4}$$

They have the following properties:

*Lemma 2.*

$$\begin{aligned}
&\psi_{\pm i}\psi_{\pm j} + q^{\pm 1}\psi_{\pm j}\psi_{\pm i} = 0 \qquad i \leqslant j \\
&\psi_i\psi_{-j} + q\psi_{-j}\psi_i = 0 \qquad \forall i \neq j \\
&\psi_n\psi_{-n} + q^2\psi_{-n}\psi_n = 0 \\
&\psi_{-i-1}\psi_{i+1} + \psi_{i+1}\psi_{-i-1} + q\psi_{-i}\psi_i + q^{-1}\psi_i\psi_{-i} = 0 \qquad i < n
\end{aligned} \tag{5}$$

and similar relations for $\phi_i$ and $\phi_{-i}$.

*Lemma 3.*

$$\begin{aligned}
&\psi_j e_i - q^{(\alpha_i, \delta_0 - \delta_j)} e_i \psi_j = \delta_{ij}\psi_{i+1} \qquad \forall i, j \\
&\psi_{-j} e_i - q^{(\alpha_i, \delta_0 + \delta_j)} e_i \psi_{-j} = \delta_{i+1, j}\psi_{-i+1} \qquad i > 1 \\
&[\psi_{i+1}, f_j\} = \delta_{ij}\psi_i K_i q_i^{-1} \\
&[\psi_{-i}, f_j\} = -\delta_{ij}\psi_{-i-1} K_i q_i^{-1}
\end{aligned} \tag{6}$$

and similar relations for $\phi_{\pm i}$, where $\psi_{n+1}$ and $\phi_{n+1}$ are understood as $\psi_{-n}$ and $\phi_{-n}$ respectively.

*Lemma 4.*

$$\{\psi_{\pm i}, \phi_{\pm i}\} = \frac{\Pi_{\pm i} - \Pi_{\pm i}^{-1}}{q - q^{-1}} \tag{7}$$

$$\{\psi_\mu, \phi_\nu\} = \begin{cases} \sum_{\hat{k},\hat{r}} c_{\hat{k},\hat{r}} K^{(\hat{r})} E^{(\hat{k})} & \mu - \nu > 0 \\ \sum_{\hat{k},\hat{r}} \tilde{c}_{\hat{k},\hat{r}} F^{(\hat{k})} K^{(\hat{r})} & \mu - \nu < 0 \end{cases} \tag{8}$$

where

$$\Pi_i = \prod_{k=0}^{i-1} K_k \qquad \Pi_{-i} = \Pi_n \prod_{k=i}^{n} K_k.$$

In (8), $\mu, \nu = \pm 1, \pm 2, \ldots, \pm n$; $c_{\hat{k},\hat{r}} \in \mathbf{C}[q, q^{-1}]$ may be non-vanishing only when $wt(E^{(\hat{k})}) = \mathrm{sign}(\nu)\delta_{|\nu|} - \mathrm{sign}(\mu)\delta_{|\mu|}$, and similarly for $\tilde{c}_{\hat{k},\hat{r}} \in \mathbf{C}[q, q^{-1}]$.

Define

$$\begin{aligned}
\bar{\Gamma}^{(\hat{\theta})} &= (\psi_1)^{\theta_1}(\psi_2)^{\theta_2}\ldots(\psi_n)^{\theta_n}(\psi_{-n})^{\theta_{-n}}(\psi_{-n+1})^{\theta_{-n+1}}\ldots(\psi_{-1})^{\theta_{-1}} \qquad \theta_{\pm i} = 0, 1 \\
\Gamma^{(\hat{\theta})} &= (\phi_{-1})^{\theta_{-1}}(\phi_{-2})^{\theta_{-2}}\ldots(\phi_{-n})^{\theta_{-n}}(\phi_n)^{\theta_n}(\phi_{n-1})^{\theta_{n-1}}\ldots(\phi_1)^{\theta_1} \qquad \theta_{\pm i} = 0, 1 \\
\bar{\Gamma} &= \bar{\Gamma}^{(\theta_{\pm i}=1, \forall i)} \\
\Gamma &= \Gamma^{(\theta_{\pm i}=1, \forall i)}.
\end{aligned}$$

*Lemma 5.* (i) Any product of $\psi$s (resp. $\phi$s) can be expressed as a linear combination of $\bar{\Gamma}^{(\hat{\theta})}$ (resp. $\Gamma^{(\hat{\theta})}$), $\hat{\theta} \in \mathbf{Z}_2^{2n}$;

(ii) $\bar{\Gamma}^{(\hat{\theta})}$ (resp. $\Gamma^{(\hat{\theta})}$), $\hat{\theta} \in \mathbf{Z}_2^{2n}$ are linearly independent over $\mathbf{C}[q, q^{-1}]$.

*Proof.* Part (i) is a direct consequence of lemma 2. To prove (ii), we note that any non-trivial relation of the form $\sum_{\hat{\theta}} c_{\hat{\theta}} \bar{\Gamma}^{(\hat{\theta})} = 0$ would lead to $\bar{\Gamma} \equiv 0$ in $U_q(C(n+1))$. Then it would follow that in any linear representation of $U_q(C(n+1))$, the $z$ defined by

$$z = \bar{\Gamma}\Gamma \tag{9}$$

vanishes identically. But it is easy to construct representations in which $z$ is non-zero.

For later use we define the following vector spaces

$$\Psi = \bigoplus_{\hat{\theta}} \mathbf{C}[q, q^{-1}]\bar{\Gamma}^{(\hat{\theta})}$$

$$\Phi = \bigoplus_{\hat{\theta}} \mathbf{C}[q, q^{-1}]\Gamma^{(\hat{\theta})}.$$

Direct computations can easily establish:

*Lemma 6.* Let $a \in U_q(sp(2n) \oplus u(1))$, and $b \in U_q(sp(2n)) \subset U_q(C(n+1))$. Then

$$\begin{aligned}
&[b, \Gamma] = 0 \\
&[b, \bar{\Gamma}] = 0 \\
&[a, z] = 0.
\end{aligned} \tag{10}$$

*Proof.* The first two equations follow from lemma 3. They also lead to the last equation.

Now we have the following generalized BPW theorem for the quantum supergroup $U_q(C(n+1))$:

*Theorem 1.* Let $U_+$ (resp. $U_-$)$\in U_q(C(n+1))$ be the subalgebra generated by $\{e_i \mid i=0,1,\ldots,n\}$ (resp. $\{f_i \mid i=0,1,\ldots,n\}$). Then

(i) $U_q(C(n+1))$ admits the triangular decomposition

$$U_q(C(n+1)) = U_- B_0 U_+ \tag{11}$$

or more precisely, the multiplication of $U_q(C(n+1))$ gives rise to the $\mathbf{C}[q,q^{-1}]$ module isomorphism

$$U_- \otimes B_0 \otimes U_+ \to U_q(C(n+1))$$

(ii) $U_+$ and $U_-$ respectively have the bases

$$\{E^{(\hat{k})}\bar{\Gamma}^{(\hat{\theta})} \mid \hat{k} \in \mathbf{Z}_+^{n^2}, \hat{\theta} \in \mathbf{Z}_2^{2n}\} \qquad \{\Gamma^{(\hat{\theta})}F^{(\hat{k})} \mid \hat{k} \in \mathbf{Z}_+^{n^2}, \hat{\theta} \in \mathbf{Z}_2^{2n}\} \tag{12}$$

(iii) The following elements form a basis for $U_q(C(n+1))$

$$\{\Gamma^{(\hat{\theta})}F^{(\hat{k})}K^{(\hat{r})}H^{(\hat{s})}E^{(\hat{l})}\bar{\Gamma}^{(\hat{\theta}')} \mid \hat{k},\hat{l} \in \mathbf{Z}_+^{n^2}, \hat{r},\hat{s} \in \mathbf{Z}_+^{n+1}, \hat{\theta},\hat{\theta}' \in \mathbf{Z}_2^{2n}\}. \tag{13}$$

*Proof.* Part (i) is a consequence of the defining relations of $U_q(C(n+1))$. $U_+$ is spanned by $\{E^{(\hat{k})}\bar{\Gamma}^{(\hat{\theta})} \mid \hat{k} \in \mathbf{Z}_+^{n^2}, \hat{\theta} \in \mathbf{Z}_2^{2n}\}$ because of lemmas 5 and 3. It follows from lemma 5 and equation (3) that these elements are linearly independent. Similarly we can show that $\{\Gamma^{(\hat{\theta})}F^{(\hat{k})} \mid \hat{k} \in \mathbf{Z}_+^{n^2}, \hat{\theta} \in \mathbf{Z}_2^{2n}\}$ forms a basis of $U_-$. (iii) follows from (i) and (ii). □

### 2.3. $U_q(C(n+1))$ *at roots of unity*

In this subsection we assume that $q$ is an $N$th primitive root of unity. We define

$$N' = \begin{cases} N & N \text{ odd} \\ N/2 & N \text{ even.} \end{cases}$$

Let $Z_q$ be the central algebra of the $\mathbf{Z}_2$-graded algebra $U_q(C(n+1))$ over the complex field $\mathbf{C}$, and let $Z_0$ be the algebra generated by the following elements

$$(K_n^{\pm})^{N'} \qquad (K_i^{\pm})^N \qquad i<n \qquad (E_{\beta_t})^N \qquad (F_{\beta_t})^N \qquad t=1,2,\ldots,n^2. \tag{14}$$

It is well known that [10] $Z_0$ is contained in the central algebra of the maximal even quantum subgroup $U_q(sp(2n)\oplus u(1))$. In fact, we also have

*Lemma 7.* $Z_q$ and $Z_0$ are as defined above. Then

$$Z_0 \subset Z_q. \tag{15}$$

*Proof.* The proofs is exactly the same as lemma 5 of [3], thus will not be repeated here.

$Z_0$ is a commutative algebra with no zero divisors. Following [10] we define the quotient field $Q(Z_0)$ of $Z_0$, and let $QU_q(C(n+1)) = Q(Z_0)\otimes_{Z_0} U_q(C(n+1))$. Then $QU_q(C(n+1))$ is finite-dimensional, with a basis

$$\{\Gamma^{(\hat{\theta})}F^{(\hat{k})}K^{(\hat{r})}E^{(\hat{l})}\bar{\Gamma}^{(\hat{\theta}')} \mid \hat{k},\hat{l} \in \mathbf{Z}_N^{n^2}; \hat{r} \in \mathbf{Z}_N^{n+1}, r_n \in \mathbf{Z}_{N'}; \hat{\theta},\hat{\theta}' \in \mathbf{Z}_2^{2n}\}. \tag{16}$$

## 3. Finite-dimensional representations

### 3.1. At generic q

Given an irreducible $U_q(sp(2n) \oplus U(1))$ module $V^{(0)}$ over $\mathbf{C}[q, q^{-1}]$, we construct a $U_q(C(n+1))$-module $\bar{V}$ over $\mathbf{C}[q, q^{-1}]$ in the following way. Impose the condition

$$e_0 v = 0 \qquad \forall v \in V^{(0)} \tag{17}$$

and construct the $\mathbf{C}[q, q^{-1}]$ module

$$\bar{V} = \Phi \bigotimes_{\mathbf{C}[q,q^{-1}]} V^{(0)}.$$

Given any element $a \in U_q(C(n+1))$, and $\Gamma^{(\hat{\theta})} \in \Phi$, it follows from theorem 1 that

$$a\Gamma^{(\hat{\theta})} = \sum c_{\hat{\theta}',\hat{k},\hat{r},\hat{s},\hat{l},\hat{\theta}''} \Gamma^{(\hat{\theta}')} F^{(\hat{k})} K^{(\hat{r})} H^{(\hat{s})} E^{(\hat{l})} \bar{\Gamma}^{(\hat{\theta}'')} \qquad c_{\hat{\theta}',\hat{k},\hat{r},\hat{s},\hat{l},\hat{\theta}''} \in \mathbf{C}[q, q^{-1}].$$

We define the action of $a$ on $\bar{V}$ by

$$a(\Gamma^{(\hat{\theta})} \otimes v) = \sum c_{\hat{\theta}',\hat{k},\hat{r},\hat{s},\hat{l},0} \Gamma^{(\hat{\theta}')} \otimes F^{(\hat{k})} K^{(\hat{r})} H^{(\hat{s})} E^{(\hat{l})} v$$

thus turning $\bar{V}$ into a $U_q(C(n+1))$-module. For simplicity, we will write $\Gamma^{(\hat{\theta})} \otimes v$ as $\Gamma^{(\hat{\theta})} v$ from here on.

Let $M$ be the maximal proper submodule contained in $\bar{V}$. Setting

$$V = \bar{V}/M \tag{18}$$

we arrive at an irreducible $U_q(C(n+1))$ module. If $M = \{0\}$, we say that $V$ and the associated irrep of $U_q(C(n+1))$ are typical, otherwise atypical.

The module $V$ has a $\mathbf{Z}_2$ gradation, i.e., $V = V_0 \oplus V_1$, with $V_0$ and $V_1$ respectively spanned by $\{\Gamma^{(\hat{\theta})} v \in V \mid [\Gamma^{(\hat{\theta})}] = 0, v \in V^{(0)}\}$, and $\{\Gamma^{(\hat{\theta})} v \in V \mid [\Gamma^{(\hat{\theta})}] = 1, v \in V^{(0)}\}$. This $\mathbf{Z}_2$ gradation is consistent with that of $U_q(C(n+1))$ itself, namely given any homogeneous element $a \in U_q(C(n+1))$, we have $aV_\epsilon \subset V_{\epsilon+[a](\mathrm{mod}2)}$, $\epsilon = 0, 1$. $V$ also has a natural $\mathbf{Z}$ gradation. Let

$$\deg(\Gamma^{(\hat{\theta})}) = \sum_{i=1}^{n} (\theta_i + \theta_{-i}) \qquad \deg(\bar{\Gamma}^{(\hat{\theta})}) = -\sum_{i=1}^{n} (\theta_i + \theta_{-i}).$$

Define $V^{(k)}$ to be the vector space spanned by $\{\Gamma^{(\hat{\theta})} v \in V \mid \deg(\Gamma^{(\theta)}) = k, v \in V^{(0)}\}$. Then

$$V = \bigoplus_{k=0}^{L} V^{(k)} \qquad L \leqslant 2n$$

with

$$\Gamma^{(\hat{\theta})} V^{(k)} \subset V^{(k+\deg(\Gamma^{(\hat{\theta})}))} \qquad V^{(l)} = \{0\} \qquad \forall l > L$$
$$\bar{\Gamma}^{(\hat{\theta})} V^{(k)} \subset V^{(k+\deg(\bar{\Gamma}^{(\hat{\theta})}))} \qquad V^{(l)} = \{0\} \qquad \forall l < 0$$

and each $V^{(k)}$ furnishes a $U_q(sp(2n) \oplus u(1))$ module,

$$aV^{(k)} \subset V^{(k)} \qquad \forall a \in U_q(sp(2n) \oplus u(1)).$$

In particular,

*Lemma 8.* (i) $V^{(L)}$ is an irreducible $U_q(sp(2n) \oplus u(1))$ module;

(ii) $L$ is equal to $2n$ if and only if $V$ is typical; and in that case, $V^{(2n)}$ and $V^{(0)}$ are isomorphic $U_q(sp(2n))$ modules.

*Proof.* (i) is required by the irreducibility of $V$. To prove the first part of (ii), we note that the necessity of $L = 2n$ is obvious. Let us assume that $L = 2n$, i.e., $\Gamma V^{(0)} \not\subset M$, but $V$ is atypical. Then there must exist at least one non-vanishing vector $u \in M$. Now we can apply elements $\phi_{\pm i}$ to $u$ to obtain another vector in $M$ of the form $\Gamma v$, for some $v \in V^{(0)}$. Since $\Gamma$ commutes with all elements of $U_q(sp(2n))$, the irreducibility of $V^{(0)}$ with respect to $U_q(sp(2n))$ implies that $\Gamma V^{(0)} \subset M$, which contradicts our assumption. The second part of (ii) follows from lemma 6.

Note that (17) is equivalent to $\psi_{\pm i} v = 0$, $\forall v \in V^{(0)}$. In any given irreducible $U_q(C(n+1))$ module, there always exists a subspace obeying this condition. Thus (17) does not impose any restrictions on the irreducible module $V$, and the construction developed above yields all irreps of the quantum supergroup $U_q(C(n+1))$.

We will call $V$ a highest weight $U_q(C(n+1))$ module if there exists a unique vector $v^\Lambda \in V$, with $\Lambda = \sum_{i=0}^{n} \lambda_i \delta_i \in H^*$, $\lambda_i \in \mathbf{C}$, such that

$$\begin{aligned} &e_i v^\Lambda = 0 \\ &\psi_{\pm i} v^\Lambda = 0 \qquad i = 1, 2, \ldots, n \\ &K_i v^\Lambda = \epsilon_i q^{(\alpha_i,\ \Lambda)} v^\Lambda \qquad \epsilon_i = \pm 1 \qquad i = 0, 1, \ldots, n. \end{aligned} \tag{19}$$

Observe that the sign factors $\epsilon_i$ may be eliminated by the following automorphism of $U_q(C(n+1))$:

$$e_i \mapsto \epsilon_i^{-1} e_i \qquad f_i \mapsto f_i \qquad K_i \mapsto \epsilon_i^{-1} K_i \qquad \forall i.$$

Hereafter we will assume that to any irreducible $U_q(C(n+1))$ module, an appropriate automorphism of this kind has been employed to cast the last equation of (19) into

$$K_i v^\Lambda = q^{(\alpha_i,\ \Lambda)} v^\Lambda \qquad i = 0, 1, \ldots, n.$$

Since it is necessarily true that $v^\Lambda \in V^{(0)}$, $V$ is a highest weight module if and only if $V^{(0)}$ is of highest weight type. To emphasize the role of the highest weight, we denote by $V^{(0)}(\Lambda)$ the $U_q(sp(2n) \oplus u(1))$ module $V^{(0)}$, and introduce the new notation $\bar{V}(\Lambda)$, $V(\Lambda)$ and $M(\Lambda)$, respectively, for the $\bar{V}$, $V$ and $M$ constructed from $V^{(0)}$.

It immediately follows from our construction that the $U_q(C(n+1))$ module $V(\Lambda)$ is finite-dimensional if and only if the associated irreducible $U_q(sp(2n) \oplus u(1))$ module $V^{(0)}(\Lambda)$ is finite-dimensional. Since a finite-dimensional irreducible $U_q(sp(2n) \oplus u(1))$ is uniquely characterized by its highest weight, so is the irreducible $U_q(C(n+1))$ module induced from it. Therefore,

*Proposition 1.* (i) The irreducible highest weight $U_q(C(n+1))$ module $V(\Lambda)$ is finite-dimensional if and only if

$$\frac{2(\alpha_i, \Lambda)}{(\alpha_i, \alpha_i)} \in \mathbf{Z}_+ \qquad i = 1, 2, \ldots, n \tag{20}$$

(ii) A finite-dimensional irreducible $U_q(C(n+1))$ module $V(\Lambda)$ is uniquely determined by its highest weight $\Lambda$.

Note that $K_i$, $i = 0, 1, \ldots, n$, are all diagonalizable on $V^{(0)}(\Lambda)$. Thus they are also diagonalizable on the entire $U_q(C(n+1))$ module $V(\Lambda)$. Define the weight space $V_\omega \subset V(\Lambda)$ to be the vector space over $\mathbf{C}[q, q^{-1}]$ consisting of all the vectors $v \in V(\Lambda)$ satisfying $K_i v = q^{(\omega, \alpha_i)} v$. Define $Sp_\Lambda$ to be the set of all the distinct $\omega$s such that $\dim_{\mathbf{C}[q,q^{-1}]} V_\omega \neq 0$. Then

$$V(\Lambda) = \oplus_{\omega \in Sp_\Lambda} V_\omega.$$

By assigning the ordering $\delta_i > \delta_j > 0$, $\forall i < j$, we achieve a partial ordering of the elements of $H^*$ with the same imaginary part. Then it is clear that $\omega \leqslant \Lambda$, and $\Lambda - \omega = \sum_{i=0}^{n} m_i\alpha_i, m_i \in \mathbf{Z}_+$.

To gain further understanding of the structures of $V(\Lambda)$, we now construct the highest weight vector of the irreducible $U_q(sp(2n) \oplus u(1))$ module $V^{(L)} \subset V(\Lambda)$. Consider the set of vectors $\{v^{(0)}, v^{(1)}, \ldots, v^{(L)}\} \subset V(\Lambda)$ defined by

$$
\begin{aligned}
&v^{(0)} = v^{\Lambda} \\
&v^{(k)} = \phi_{\mu_k} v^{(k-1)} \neq 0 \\
&\phi_v v^{(k-1)} = 0 \qquad \text{if } wt(\phi_v) > wt(\phi_{\mu_k})
\end{aligned}
$$

where $\mu_k, v = \pm 1, \pm 2, \ldots, \pm n$. Since $\phi_i$, $i = 1, 2, \ldots, n$, all $q$-anticommute, the existence of the vectors $\{v^{(0)}, v^{(1)}, \ldots, v^{(l)}\}$ with $\mu_t > 0$, $\forall t = 1, 2, \ldots, l$, is guaranteed, where $v^{(l)}$ is annihilated by all $\phi_i$. Now if $v^{(l)}$ is also annihilated by all $\phi_{-i}$, then $l = L$; otherwise there must exist a $\phi_{\mu_{l+1}}$, $\mu_{l+1} < 0$ which does not annihilated this vector, but all $\phi_{-i}$ with $wt(\phi_{-i}) > wt(\phi_{\mu_{l+1}})$ do. We set $v^{(l+1)} = \phi_{\mu_{l+1}} v^{(l)}$. Using lemma 2 we can easily see that

$$
\begin{aligned}
&\phi_i v^{(l+1)} = 0 \qquad \forall i \\
&\phi_{-i} v^{(l+1)} = 0 \qquad i > -\mu_{l+1}.
\end{aligned}
$$

Continue this process we will eventually arrive at $v^{(L)}$. It follows from the construction that

*Lemma 9.* All $v^{(k)}$, $k = 0, 1, \ldots, L$, are $U_q(sp(2n) \oplus u(1))$ highest weight vectors.

In particular, $v^{(L)}$ is the highest weight vector of the irreducible $U_q(sp(2n) \oplus u(1))$ module $V^{(L)} \subset V(\Lambda)$. If $V(\Lambda)$ is typical, then $v^{(L)} = \Gamma v^{\Lambda}$ can be raised back to $v^{\Lambda}$ by the action of $\bar{\Gamma}$. Using the above lemma and lemma 4 we can compute

$$
\begin{aligned}
&\bar{\Gamma}\Gamma v^{\Lambda} = z(\Lambda) v^{\Lambda} \\
&z(\Lambda) = \prod_{\gamma \in \Delta_1^+} \frac{q^{(\Lambda+\rho,\gamma)} - q^{-(\Lambda+\rho,\gamma)}}{q - q^{-1}}.
\end{aligned}
\tag{21}
$$

Therefore,

*Proposition 2.* The irreducible highest weight $U_q(C(n+1))$ module $V(\Lambda)$ is typical if and only if

$$
z(\Lambda) \neq 0 \tag{22}
$$

where $z(\Lambda)$ is defined by (21).

On any irreducible highest weight $U_q(C(n+1))$ module $V(\Lambda)$ with a real highest weight $\Lambda$, we introduce a sesquilinear form $\langle . \mid . \rangle : V(\Lambda) \otimes V(\Lambda) \to \mathbf{C}[q, q^{-1}]$, which satisfies the following defining relations:

(i) If $v^{\Lambda} \in V(\Lambda)$ is the highest weight vector,

$$
\langle v^{\Lambda} \mid v^{\Lambda} \rangle = 1
$$

(ii)

$$
\langle u \mid av \rangle = \langle \omega(a) u \mid v \rangle \qquad \forall a \in U_q(C(n+1)), u, v \in V(\Lambda)
$$

where $\omega$ is the anti-automorphism defined before;

(iii)

$$
\begin{aligned}
&\langle c_1 u_1 + c_2 u_2 \mid v \rangle = c_1^* \langle u_1 \mid v \rangle + c_2^* \langle u_2 \mid v \rangle \\
&\langle v \mid c_1 u_1 + c_2 u_2 \rangle = c_1 \langle v \mid u_1 \rangle + c_2 \langle v \mid u_2 \rangle
\end{aligned}
$$

where $c_1, c_2 \in \mathbf{C}[q, q^{-1}]$, $c_i^* = \omega(c_i)$, $u_1, u_2, v \in V(\Lambda)$. Note that this form is well defined as long as the highest weight is real, and has the standard property $\langle u \mid v\rangle = (\langle u \mid v\rangle)^*, \forall u, v \in V(\Lambda)$. Also,

*Lemma 10.* The form $\langle . \mid . \rangle : V(\Lambda) \otimes V(\Lambda) \to \mathbf{C}[q, q^{-1}]$ is non-degenerate.

*Proof.* The proof is rather straightforward, we nevertheless present it here. Let Ker $\subset V(\Lambda)$ be the maximal subspace such that for any $k \in$ Ker, $\langle v \mid k\rangle = 0, \forall v \in V(\Lambda)$. Then $\langle v \mid ak\rangle = \langle \omega(a)v \mid k\rangle = 0, \forall a \in U_q(C(n+1)), v \in V(\Lambda)$, i.e., Ker is an invariant subspace. Therefore we must have Ker $= \{0\}$ as required by the irreducibility of $V(\Lambda)$.

Now we compute the value of $\langle v^{(L)} \mid v^{(L)}\rangle$, which is non-vanishing as required by the the non-degeneracy of the form. As $v^{(k)}$ are $U_q(sp(2n) \oplus u(1))$ maximal vectors, it follows from lemma 2 that $\psi_{\mu_k} v^{(k-1)} = 0$. Thus

$$\langle v^{(k)} \mid v^{(k)}\rangle = \left\langle v^{(k-1)} \,\middle|\, \frac{\Pi_{\mu_k} - \Pi_{\mu_k}^{-1}}{q - q^{-1}} v^{(k-1)} \right\rangle.$$

Therefore,

$$\langle v^{(L)} \mid v^{(L)}\rangle = \prod_{k=1}^{L} \frac{q^{(\delta_0 - \delta_{\mu_k}, \Lambda + \sum_{t=1}^{k}[\delta_0 - \delta_{\mu_t}])} - q^{-(\delta_0 - \delta_{\mu_k}, \Lambda + \sum_{t=1}^{k}[\delta_0 - \delta_{\mu_t}])}}{q - q^{-1}}. \tag{23}$$

Since $\langle v^{(L)} \mid v^{(L)}\rangle \neq 0$, we have

$$\prod_{k=1}^{L}\left(\delta_0 - \delta_{\mu_k}, \Lambda + \sum_{t=1}^{k}[\delta_0 - \delta_{\mu_t}]\right) \neq 0. \tag{24}$$

Let $\mathbf{I} \subset \mathbf{C}[q, q^{-1}]$ be the ideal generated by $q - 1$. It is clear that $\mathbf{C} = \mathbf{C}[q, q^{-1}]/\mathbf{I}$. We define $\tilde{V}(\Lambda) = \{\mathbf{C}[q, q^{-1}]/\mathbf{I}\} \otimes V(\Lambda)$, and $\tilde{V}_\omega = \{\mathbf{C}[q, q^{-1}]/\mathbf{I}\} \otimes V_\omega$ for any weight space $V_\omega \subset V(\Lambda)$. Then

$$\begin{aligned} &\dim_{\mathbf{C}} \tilde{V}(\Lambda) = \dim_{\mathbf{C}[q,q^{-1}]} V(\Lambda) \\ &\tilde{V}(\Lambda) = \oplus_{\omega \in Sp_\Lambda} \tilde{V}_\omega. \end{aligned}$$

Denote by $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{h}_i$, and 1 respectively the endomorphisms on $\tilde{V}(\Lambda)$ defined by the $V(\Lambda)$ endomorphisms $e_i$, $f_i$, $(K_i - K_i^{-1})/(q_i - q_i^{-1})$, and $K_i^{\pm 1}$ through extension of scalars. It can be proved that

*Lemma 11.* The $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{h}_i$, $i = 0, 1, \ldots, n$, satisfy the defining relations of the Lie superalgebra $C(n+1)$. Thus $\tilde{V}(\Lambda)$ furnishes a $U(C(n+1))$ module.

In particular, $v^\Lambda$ remains to be a highest weight vector in $\tilde{V}(\Lambda)$. Repeatedly applying the $\tilde{f}_i$'s to it generates the entire $U(C(n+1)$ module $\tilde{V}(\Lambda)$. Therefore, $\tilde{V}(\Lambda)$ is indecomposable. It immediately follows that

*Proposition 3.* The $U(C(n+1)$ module $\tilde{V}(\Lambda)$ is typical and irreducible if and only if the $U_q(C(n+1))$ module $V(\Lambda)$ is typical.

When the highest weight $\Lambda$ is real, we denote the restriction of the form $\langle . \mid . \rangle$ on $\tilde{V}(\Lambda)$ by $\langle . \mid . \rangle_0$, which maps $\tilde{V}(\Lambda) \otimes_{\mathbf{C}} \tilde{V}(\Lambda)$ to $\mathbf{C}$. Then $\langle . \mid . \rangle_0$ satisfies similar properties as (1)–(3). Furthermore,

*Proposition 4.* The form $\langle . \mid . \rangle_0 : \tilde{V}(\Lambda) \otimes_{\mathbf{C}} \tilde{V}(\Lambda) \to \mathbf{C}$, is non-degenerate.

*Proof.* Since the $U(C(n+1))$ module $\tilde{V}(\Lambda)$ is indecomposable, for every non-vanishing $u \in \tilde{V}(\Lambda)$ there exists at least one element $\tilde{\phi} \in U(C(n+1))$ which is a product of some $\tilde{\phi}_i$s (if $u \in \tilde{V}^{(L)}(\Lambda)$, then $\tilde{\phi} = 1$), such that the vector $v = \tilde{\phi}u \neq 0$, and $v \in \tilde{V}^{(L)}(\Lambda)$. If the restriction of $\langle . \mid . \rangle_0$ on $\tilde{V}^{(L)}(\Lambda)$ is non-degenerate, then $\langle v' \mid v \rangle_0$ does not vanish for some elements $v' \in \tilde{V}^{(L)}(\Lambda)$. Now

$$\langle \tilde{\omega}(\tilde{\phi})v' \mid u \rangle_0 = \langle v' \mid \tilde{\phi}u \rangle_0 \neq 0$$

where $\tilde{\omega}$ is the $q \to 1$ limit of the anti-automorphism $\omega$. Therefore the form $\langle . \mid . \rangle_0$ cannot be degenerate on $\tilde{V}(\Lambda)$. The converse is also obviously true, thus we conclude that $\langle . \mid . \rangle_0$ is non-degenerate if and only if it is non-degenerate on $\tilde{V}^{(L)}(\Lambda)$.

It follows from the theorem of Lusztig and Rosso [7] that $\tilde{V}^{(L)}(\Lambda)$ is an irreducible $sp(2n) \oplus u(1)$ module, with the highest weight vector $\tilde{v}^{(L)}$ which is the $q \to 1$ limit of $v^{(L)}$. Therefore $\langle . \mid . \rangle_0$ will be non-degenerate on $\tilde{V}^{(L)}(\Lambda)$ if $\langle \tilde{v}^{(L)} \mid \tilde{v}^{(L)} \rangle_0 \neq 0$. This is indeed that case, as it follows from (23) and (24) that

$$\langle \tilde{v}^{(L)} \mid \tilde{v}^{(L)} \rangle_0 = \prod_{k=1}^{L} \left( \delta_0 - \delta_{\mu_k}, \Lambda + \sum_{t=1}^{k} [\delta_0 - \delta_{\mu_t}] \right) \neq 0.$$

The non-degeneracy of $\langle . \mid . \rangle_0$ implies that the indecomposible $U(C(n+1))$ module $\tilde{V}(\Lambda)$ is irreducible. To see this, we note that if $\tilde{V}(\Lambda)$ was reducible, then there must exist at least one $u \in \tilde{V}(\Lambda)$ which could not be mapped to the highest weight vector $v^\Lambda$, or equivalently

$$\langle v^\Lambda \mid \tilde{a}u \rangle_0 = 0 \qquad \forall \tilde{a} \in U(C(n+1)).$$

This would lead to

$$\langle \tilde{\omega}(\tilde{a})v^\Lambda \mid u \rangle_0 = 0 \qquad \forall \tilde{a} \in U(C(n+1)). \tag{25}$$

$\tilde{V}(\Lambda)$ being an indecomposible $U(C(n+1))$ module, every element of it can be expressed as $\tilde{a}v^\Lambda$, $\tilde{a} \in U(C(n+1))$. Thus equation (25) would imply the degeneracy of $\langle . \mid . \rangle_0$.

Combining the above discussion with proposition 4, we arrive at the following

*Theorem 2.* Let $V(\Lambda)$ be an irreducible $U_q(C(n+1))$ module with an integrable dominant highest weight $\Lambda$(i.e., satisfying (20)), and $\tilde{V}(\Lambda)$ be as defined before. Then $\tilde{V}(\Lambda)$ is an irreducible $U(C(n+1))$ module which has the same weight space decomposition as $V(\Lambda)$.

*Remarks.*

(i) The form $\langle . \mid . \rangle_0$ can be defined independently of $\langle . \mid . \rangle$.

(ii) The proof of Theorem 2 makes essential use of (24), which can be proved without resorting to the form $\langle . \mid . \rangle$.

(iii) The forms $\langle . \mid . \rangle$ and $\langle . \mid . \rangle_0$ are merely employed to make the proof of theorem 2 more coherent; they can be avoided entirely.

Define the formal character of a finite-dimensional irreducible $U_q(C(n+1))$ module $V(\Lambda)$ by

$$\mathrm{ch}_{V(\Lambda)} = \sum_{\omega \in Sp_\Lambda} \dim_{\mathbb{C}[q,q^{-1}]} V_\omega e^\omega.$$

Using theorem 2 and the results of [8], we obtain:

*Theorem 3.* Let $V(\Lambda)$ be an irreducible $U_q(C(n+1))$ module with an integrable dominant highest weight $\Lambda$. Then

$$ch_{V(\Lambda)} = \frac{\sum_{\sigma \in W} \det(\sigma) \mathrm{e}^{\sigma(\Lambda+\rho_0)} \prod_{\gamma \in \Delta_1^+(\Lambda)} (1 + \mathrm{e}^{-\sigma(\gamma)})}{\prod_{\alpha \in \Delta_0^+} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2})} \tag{26}$$

where $W$ represents the Weyl group of the $sp(2n) \subset C(n+1)$ subalgebra, and

$$\Delta_1^+(\Lambda) = \begin{cases} \Delta_1^+ & \text{if } (\Lambda+\rho,\gamma) \neq 0 \qquad \forall\gamma \in \Delta_1^+ \\ \Delta_1^+ - \gamma_a & \text{if } \exists\gamma_a \in \Delta_1^+ \text{ such that } (\Lambda+\rho,\gamma_a) = 0. \end{cases}$$

It should be noted that when $\Lambda \in H^*$ is integrable dominant, there can exist at most one odd root $\gamma_a \in \Delta_1^+$ rendering $(\Lambda+\rho,\gamma_a) = 0$, i.e., no two factors in the product expression (21) of $z(\Lambda)$ can vanish simultaneously. Adopting the terminology of the representation theory of Lie superalgebras, we say that a finite-dimensional irrep of $U_q(C(n+1))$ at generic $q$ is either typical or singly atypical. We will see in the next subsection that this is no longer true when $q$ is a root of unity.

### 3.2. *At roots of unity*

The method developed in the last subsection for constructing $U_q(C(n+1))$ irreps works equally well when $q$ is a root of unity. Because of (16), an irreducible $U_q(sp(2n) \oplus u(1))$ module $V^{(0)}$ over $\mathbf{C}$ is necessarily finite-dimensional, thus we conclude that all irreps of $U_q(C(n+1))$ at roots of unity are finite-dimensional.

Properties of the irreducible $U_q(C(n+1))$ module induced from $V^{(0)}$ are completely determined by those of $V^{(0)}$, while $V^{(0)}$ itself is uniquely characterized by a set of complex parameters associated with the eigenvalues of the generators of $Z_0$. We say that $V$ is cyclic if the eigenvalues of all the $(E_{\beta_t})^N$ and $(F_{\beta_t})^N$ are non-vanishing, semicyclic if some are non-vanishing, and of highest weight type otherwise.

Typicality of $V$ is defined in exactly the same way as in the case with generic $q$. $V$ is typical if and only if the eigenvalue $z_V$ of $z$ defined by (9) on $V^{(0)}$ is not zero. We have

$$\dim_{\mathbf{C}} V = 2^{2n} \dim_{\mathbf{C}} V^{(0)} \qquad \text{if } z_V \neq 0.$$

When $V$ is a highest weight module, there exists a $v_0 \in V$ such that

$$K_i v_0 = a_i v_0$$
$$e_i v_0 = 0 \qquad i = 0, 1, \ldots, n$$

where $a_i$s are complex parameters. Then the eigenvalue $z_V$ of $z$ is given by

$$z_V = \prod_{\gamma\in\Delta_1^+} \frac{\pi_\gamma q^{(\rho,\gamma)} - \pi_\gamma^{-1} q^{-(\rho,\gamma)}}{q - q^{-1}}$$

$$\pi_{\delta_0-\delta_i} = \prod_{k=0}^{i-1} a_k$$

$$\pi_{\delta_0+\delta_i} = \pi_{\delta_0-\delta_n} \prod_{k=i}^{n} a_k.$$

Following the convention of the representation theory of Lie superalgebras, we call an atypical $U_q(C(n+1))$ module $V$ singly atypical if only one factor in $z_V$ is zero, and multiply atypical otherwise. There exist $a_i$ values rendering $V$ multiply atypical. Therefore, $U_q(C(n+1))$ *admits (semi)cyclic irreps and multiply atypical irreps at roots of unity*. In contrast, all finite dimensional irreps of $U_q(C(n+1))$ at generic $q$ are of highest weight type, and either typical or singly atypical.

## 4. Irreducible representations of $U_q(C(2))$

In this section we construct all the highest weight irreps of the quantum supergroup $U_q(C(2))$ at generic $q$ and all the irreps at roots of unity. For convenience, we change the notation from the general case by letting

$$\psi_+ = \psi_1 \qquad \psi_- = \psi_{-1}$$
$$\phi_+ = \phi_1 \qquad \phi_- = \phi_{-1}$$
$$e = e_1 \qquad f = f_1.$$

When the deformation parameter $q$ is generic, $U_q(C(2))$ has the following basis

$$\{(\phi_-)^{\theta_-}(\phi_+)^{\theta_+} f^k K^{(\hat{r})} H^{(\hat{s})} e^l (\psi_+)^{\theta'_+}(\psi_-)^{\theta'_-} \mid k, l \in \mathbf{Z}_+,\ \hat{r}, \hat{s} \in \mathbf{Z}_+^2,\ \theta_\pm, \theta'_\pm \in \{0, 1\}\}.$$

Let $V(\Lambda)$ be an irreducible $U_q(C(2))$ module with highest weight $\Lambda = \lambda_0\delta_0 + \lambda_1\delta_1$, and maximal vector $v^\Lambda$. It is finite dimensional if and only if $\lambda_1 \in \mathbf{Z}_+$, and in that case, $\Lambda$ must satisfy one of the following three mutually exclusive conditions:

(i) $(\Lambda + \rho, \gamma) \neq 0 \qquad \forall \gamma \in \Delta_1^+$

(ii) $(\Lambda + \rho, \delta_0 - \delta_1) = 0$

(iii) $(\Lambda + \rho, \delta_0 + \delta_1) = 0$.

We explicitly construct $V(\Lambda)$ for all the cases below:

(i) $(\Lambda + \rho, \gamma) \neq 0\ \forall \gamma \in \Delta_1^+$;

$$V(\Lambda) = \bigoplus_{\substack{\theta_\pm \in \{0,1\} \\ i \in \{0,1,\ldots,\lambda_1\}}} \mathbf{C}[q, q^{-1}]\phi_-^{\theta_-}\phi_+^{\theta_+} f^i v^\Lambda.$$

(ii) $(\Lambda + \rho, \delta_0 - \delta_1) = 0$;

$$V(\Lambda) = \begin{cases} \bigoplus_{i \in \{0,1,\ldots,\lambda_1\}} \mathbf{C}[q, q^{-1}] f^i v^\Lambda \quad \bigoplus_{j \in \{0,1,\ldots,\lambda_1 - 1\}} \mathbf{C}[q, q^{-1}] f^j \phi_- v^\Lambda & \lambda_1 \neq 0 \\ \mathbf{C}[q, q^{-1}] v^\Lambda & \lambda_1 = 0. \end{cases}$$

(iii) $(\Lambda + \rho, \delta_0 + \delta_1) = 0$;

$$V(\Lambda) = \bigoplus_{i \in \{0,1,\ldots,\lambda_1\}} \mathbf{C}[q, q^{-1}] f^i v^\Lambda \quad \bigoplus_{j \in \{0,1,\ldots,\lambda_1+1\}} \mathbf{C}[q, q^{-1}] f^j \phi_+ v^\Lambda.$$

When $\lambda_1 \notin \mathbf{Z}_+$, $V(\Lambda)$ is infinite-dimensional. Then $V(\Lambda)$ belongs to one of the following three cases:

(i) $(\Lambda + \rho, \gamma) \neq 0\ \forall \gamma \in \Delta_1^+$;

$$V(\Lambda) = \bigoplus_{\substack{\theta_\pm \in \{0,1\} \\ i \in \mathbf{Z}_+}} \mathbf{C}[q, q^{-1}]\phi_-^{\theta_-}\phi_+^{\theta_+} f^i v^\Lambda.$$

(ii) $(\Lambda + \rho, \delta_0 + \delta_1) = 0 \qquad (\Lambda + \rho, \delta_0 - \delta_1) \neq 0$;

$$V(\Lambda) = \bigoplus_{\substack{\theta \in \{0,1\} \\ i \in \mathbf{Z}_+}} \mathbf{C}[q, q^{-1}] f^i \phi_+^\theta v^\Lambda.$$

(iii) $(\Lambda + \rho, \delta_0 - \delta_1) = 0$;

$$V(\Lambda) = \bigoplus_{\substack{\theta \in \{0,1\} \\ i \in \mathbf{Z}_+}} \mathbf{C}[q, q^{-1}] f^i \phi_-^\theta v^\Lambda.$$

It is interesting to observe that in all the three cases with $\lambda_1 \notin \mathbf{Z}_+$, $V(\Lambda)$ has finite-dimensional weight spaces. In the limit $q \to 1$, $V(\Lambda)$ reduces to an infinite-dimensional irreducible $U(C(2))$ module, which has the same weight space decomposition as $V(\Lambda)$ itself.

Now we assume that $q$ is an $N''$th primitive root of unity. Let

$$N' = \begin{cases} N'' & N'' \text{ odd} \\ N''/2 & N'' \text{ even} \end{cases} \qquad N = \begin{cases} N' & N' \text{ odd} \\ N'/2 & N' \text{ even.} \end{cases}$$

Then the following elements are all in the centre of $U_q(C(2))$:

$$(K_0^{\pm})^{N''} \qquad (K_1^{\pm})^{N'} \qquad e^{N'} \qquad f^{N'}$$

provided $N' > 1$.

We will call an irrep of $U_q(C(2))$ a highest weight irrep if it possesses both a highest and lowest weight vector, or equivalently,

$$\det(e) = \det(f) = 0.$$

Such an irrep furnished by the irreducible module $V(a_0, a_1)$ is uniquely determined by the two complex parameters $a_0,\ a_1$ defined in the following way: let $v_+$ be the highest weight vector of $V(a_0, a_1)$, then

$$K_0 v_+ = a_0 v_+ \qquad K_1 v_+ = a_1 v_+. \tag{27}$$

We further define

$$d = \begin{cases} i & \text{if } a_1 = \pm q^{-2(i-1)}, \text{ with } N \geqslant i \geqslant 1, \\ N & \text{otherwise} \end{cases}$$

$$\tilde{d} = \begin{cases} d & \text{if } a_1 = \pm q^{-2(d-1)} \\ d-1 & \text{otherwise.} \end{cases}$$

$$V^{(0)}(a_0, a_1) = \bigoplus_{i=0}^{d} \mathrm{C} f^i v_+.$$

$V(a_0, a_1)$ can only belong to one of the following four cases:

(i) $a_0 - a_0^{-1} \neq 0 \qquad a_0 a_1 q^{-2} - a_0^{-1} a_1^{-1} q^2 \neq 0$;

$$V(a_0, a_1) = V^{(0)}(a_0, a_1) \bigoplus V^{(1)}(a_0, a_1) \bigoplus V^{(2)}(a_0, a_1)$$

$$V^{(1)}(a_0, a_1) = \bigoplus_{i=0}^{d-1} \mathrm{C}\phi_+ f^i v_+ \bigoplus \{\bigoplus_{i=0}^{d-1} \mathrm{C}\phi_- f^i v_+\}$$

$$V^{(2)}(a_0, a_1) = \bigoplus_{i=0}^{d-1} \mathrm{C}\phi_+\phi_- f^i v_+.$$

(ii) $a_0 - a_0^{-1} = 0 \qquad a_0 a_1 q^{-2} - a_0^{-1} a_1^{-1} q^2 \neq 0$;

$$V(a_0, a_1) = V^{(0)}(a_0, a_1) \bigoplus V^{(1)}(a_0, a_1)$$

$$V^{(1)}(a_0, a_1) = \bigoplus_{i=0}^{\tilde{d}-1} \mathrm{C}\phi_- f^i v_+.$$

(iii) $a_0 - a_0^{-1} \neq 0 \qquad a_0 a_1 q^{-2} - a_0^{-1} a_1^{-1} q^2 = 0$;

$$V(a_0, a_1) = V^{(0)}(a_0, a_1) \bigoplus V^{(1)}(a_0, a_1)$$

$$V^{(1)}(a_0, a_1) = \mathrm{C}\phi_+ v_+ \bigoplus \{\bigoplus_{i=0}^{\tilde{d}-1} \mathrm{C}\phi_- f^i v_+\}.$$

(iv) $a_0 - a_0^{-1} = a_0 a_1 q^{-2} - a_0^{-1} a_1^{-1} q^2 = 0$;

$$V(a_0, a_1) = V^{(0)}(a_0, a_1) \bigoplus V^{(1)}(a_0, a_1)$$

$$V^{(1)}(a_0, a_1) = \bigoplus_{i=0}^{d-2} \mathbf{C}\phi_- f^i v_+.$$

Having explicitly constructed the highest weight irreps of $U_q(C(2))$, we now consider the (semi)cyclic irreps. We start with the simpler case that $N''$ is not divisible by 4. The (semi)cyclic irreducible module $V^{(0)}$ over the maximal even subalgebra $U_q(sp(2) \oplus u(1))$ is $N$-dimensional, and labelled by four parameters. Explicitly, we have the following basis $\{v_i \mid i = 0, 1, \ldots, N\}$ for $V^{(0)}$, with the actions of the generators of $U_q(sp(2) \oplus u(1))$ defined by

$$\begin{aligned} &K_0 v_0 = a_0 \qquad && K_1 v_0 = a_1 v_0 \\ &e_1 v_0 = x v_{N-1} \qquad && f_1 v_{N-1} = x' v_0 \\ &f_1 v_i = v_{i+1} \qquad && i = 0, 1, \ldots, N-2 \end{aligned} \tag{28}$$

where the complex parameters $x$ and $x'$ do not vanish simultaneously, and

$$xx' \neq \frac{(q^{2i} - q^{-2i})(a_1 q^{2(i-1)} - a_1^{-1} q^{-2(i-1)})}{q^2 - q^{-2}} \qquad i = 1, 2, \ldots, N-1. \tag{29}$$

For simplicity, we introduce the new parametrization

$$a_1 = q^2 b b' \qquad x = \frac{u(b - b^{-1})}{q^2 - q^{-2}} \qquad x' = -\frac{u^{-1}(b' - b'^{-1})}{q^2 - q^{-2}} \tag{30}$$

and also define

$$Q = \frac{(a_0 b - a_0^{-1} b^{-1})(a_0 b' - a_0^{-1} b'^{-1})}{(q - q^{-1})^2}. \tag{31}$$

Denote by $V$ the irreducible (semi)cyclic $U_q(C(2))$ module induced from $V^{(0)}$. Then

(i) If $Q \neq 0$,

$$V = \bigoplus_{l=0}^{2} V^{(l)}$$

$$V^{(1)} = \bigoplus_{i=0}^{N-1} \{\mathbf{C}\phi_+ v_i \oplus \mathbf{C}\phi_- v_i\}$$

$$V^{(2)} = \bigoplus_{i=0}^{N-1} \mathbf{C}\phi_- \phi_+ v_i$$

(ii) If $Q = 0$, but either $x' \neq 0$ or $a_0 b - a_0^{-1} b^{-1} \neq 0$,

$$V = V^{(0)} \bigoplus V^{(1)}$$

$$V^{(1)} = \bigoplus_{i=0}^{N-1} \mathbf{C}\phi_- v_i$$

(iii) If $Q = 0$, $x' = 0$, $a_0 b - a_0^{-1} b^{-1} = 0$,

$$V = V^{(0)} \bigoplus V^{(1)}$$

$$V^{(1)} = \bigoplus_{i=0}^{N-2} \mathbf{C}\phi_- v_i.$$

When $N''$ is divisible by four, the (semi)cyclic irreducible module $V^{(0)}$ over the maximal even subalgebra $U_q(sp(2) \oplus u(1))$ is $2N$-dimensional, with a basis $\{v_i^{(\pm)} \mid i = 0, 1, \ldots, N-1\}$ such that

$$\begin{aligned} &e_1 v_0^{(\pm)} = \pm x v_{N-1}^{(\pm)} \qquad && f_1 v_{N-1}^{(\pm)} = \pm x' v_0^{(\pm)} \\ &f_1 v_i^{(\pm)} = v_{i+1}^{(\pm)} && i = 0, 1, \ldots, N-2 \\ &K_1 v_0^{(\pm)} = a_1 v_0^{(\pm)} && K_0 v_0^{(\pm)} = a_0 v^{(\mp)} \end{aligned} \tag{32}$$

where again $x$ and $x'$ do not vanish simultaneously, and obey the constraint (29). Note that from this basis we can always obtain a new one in which $K_0^{(\pm 1)}$ are diagonal. Let

$$V^{(0)} = V_+^{(0)} \bigoplus V_-^{(0)} \qquad V_\pm^{(0)} = \bigoplus_{i=0}^{N-1} \mathbf{C} v_i^{(\pm)}.$$

Then the irreducible $U_q(sp(2) \oplus u(1))$ module $V$ induced from $V^{(0)}$ is given by

(i) If $Q \neq 0$,

$$V = \bigoplus_{l=0}^{2} \{V_+^{(l)} \bigoplus V_-^{(l)}\}$$

$$V_\pm^{(1)} = \bigoplus_{i=0}^{N-1} \{\mathbf{C}\phi_+ v_i^{(\pm)} \oplus \mathbf{C}\phi_- v_i^{(\pm)}\}$$

$$V_\pm^{(2)} = \bigoplus_{i=0}^{N-1} \mathbf{C}\phi_-\phi_+ v_i^{(\pm)}$$

(ii) If $Q = 0$, but either $x' \neq 0$ or $a_0 b - a_0^{-1} b^{-1} \neq 0$,

$$V = \bigoplus_{\sigma=+,-} \{V_\sigma^{(0)} \bigoplus V_\sigma^{(1)}\}$$

$$V_\pm^{(1)} = \bigoplus_{i=0}^{N-1} \mathbf{C}\phi_- v_i^{(\pm)}$$

(iii) If $Q = 0$, $x' = 0$, $a_0 b - a_0^{-1} b^{-1} = 0$,

$$V = \bigoplus_{\sigma=+,-} \{V_\sigma^{(0)} \bigoplus V_\sigma^{(1)}\}$$

$$V_\pm^{(1)} = \bigoplus_{i=0}^{N-2} \mathbf{C}\phi_- v_i^{(\pm)}.$$

## 5. Conclusion

We have presented a systematic treatment of the representation theory of the quantum supergroup $U_q(C(n+1))$. The induced module construction developed here allows the irreps of this quantum supergroup at arbitrary $q$ to be constructed. Structures of the finite dimensional irreps at generic $q$ have been investigated. In particular, it has been shown that every such irrep is a deformation of an irrep of the underlying Lie superalgebra $C(n+1)$. The character formula for the finite-dimensional irreps of $U_q(C(n+1))$ is given.

We have also shown that when $q$ is a root of unity, all irreps of $U_q(C(n+1))$ are finite dimensional, and (semi)cyclic irreps also exist. The typicality criterion for highest weight irreps are given. The structures of the typicals are understood, and the general framework has also be set up for analysing the structures of the atypicals. However, further investigation into this problem will necessarily require detailed knowledge of the irreps of the maximal even subalgebra $U_q(sp(2n) \oplus u(1))$ at roots of unity.

Reference [3] and the present paper essentially complete the representation theory for the type-I quantum supergroups at generic $q$. With certain modifications, the techniques developed in these papers can also be generalized to systematically study the representation theory of the type-II quantum supergroups. Results will be reported in a forthcoming publication.

## References

[1] Kulish P P and Reshetikhin N Yu 1989 *Lett. Math. Phys.* **18** 143
Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72
Bracken A J, Gould M D and Zhang R B 1990 *Mod. Phys. Lett.* A **5** 831
Tolstoy V N 1990 *Springer Lecture Notes in Physics 370* (Berlin: Springer) p 118
Floreanini R, Spiridonov V P and Vinet L 1991 *Commun. Math. Phys.* **137** 149
Scheunert M 1992 *Lett. Math. Phys.* **24** 173
Floreanini R, Leites D A and Vinet L 1991 *Lett. Math. Phys.* **23** 127
[2] Kac V G 1977 *Adv. Math.* **26** 8
Scheunert M 1979 *The Theory of Lie Superalgebras (Springer Lecture Notes in Mathematics 716)* (Berlin: Springer)
[3] Zhang R B 1993 *J. Math. Phys.* **34** 1236
[4] Palev C D 1991 and Tolstoy V N 1991 *Commun. Math. Phys.* **141** 599
[5] Zhang R B 1992 *Lett. Math. Phys.* **25** 317
[6] Perk J H H and Schultz C L 1981 *Phys. Lett.* **84A** 407
Bazhanov V V and Shadrikov A G 1987 *Theor. Math. Phys.* **73** 1302
[7] Lusztig G 1988 *Adv. Math.* **70** 237
Rosso M 1988 *Commun. Math. Phys.* **117** 581
[8] Van de Jeugt J 1991 *Commun. Algebra* **19** 199
[9] Lusztig G 1990 *Geometriae Dedicata* **35** 89
[10] De Concini C and Kac V G 1990 Representations of quantum groups at roots of unity *Operator Algebras in Unitary Representations, Enveloping Algebras and Invariant Theory* ed A Connes *et al* *Prog. Math.* **92** 471